Page No. GOPI
Date / /

MEET KHSLANA

CSE "Core"

17

Assignment: 03

Assignment Based on Unit-III
(Differential Calculus - Function of several variables.)

1. Find the first order derivative of $\log(x^2+y^2)$

Given the function $f(x,y) = \log(x^2+y^2)$

Partial derivative w.r.t x.

Using chain rule, differentiate $\log(u)$ w.r.t. u.

$u = x^2+y^2$

$$\frac{\partial}{\partial x}\log(x^2+y^2) = \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial x}(x^2+y^2)$$

Now, differentiating x^2+y^2 w.r.t. x

$$\frac{\partial}{\partial x}(x^2+y^2) = 2x$$

So, partial derivative w.r.t. x is

$$\frac{\partial}{\partial x}\log(x^2+y^2) = \frac{2x}{x^2+y^2}$$

Partial derivative w.r.t. y

applying chain rule for y

$$\frac{\partial}{\partial y}\log(x^2+y^2) = \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial y}(x^2+y^2)$$

Now differentiating x^2+y^2 w.r.to. y.

$$\frac{\partial}{\partial y}(x^2+y^2) = 2y$$

So, partial derivative w.r.t. y is

$$\frac{\partial}{\partial y}\log(x^2+y^2) = \frac{2y}{x^2+y^2}$$

$\therefore$ First order partial derivative of $\log(x^2+y^2)$

$$\frac{\partial}{\partial x}\log(x^2+y^2) = \frac{2x}{x^2+y^2}$$

$$\frac{\partial}{\partial x}\log(x^2+y^2) = \frac{2y}{x^2+y^2}$$

2. If $z = x^3 + y^3 - 3axy$ then prove that
$\delta^2 z / \delta y \delta x = \delta^2 z / \delta y \delta x$.

Given,

$$z = x^3 + y^3 - 3axy$$

We'll find the second mixed partial derivatives in both order.

1. finding $\frac{\partial z}{\partial x}$

partial derivative of z w.r.t. x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(3axy)$$

$$\frac{\partial z}{\partial x} = 3x^2 + 0 - 3ay$$

$$\frac{\partial z}{\partial x} = 3x^2 - 3ay$$

2. Now finding $\frac{\partial^2 z}{\partial y \partial x}$

differentiating $\frac{\partial z}{\partial x}$ w.r.t y

$$\frac{\partial}{\partial y}(3x^2 - 3ay) = 0 - 3a$$

so, $$\frac{\partial^2 z}{\partial y \partial x} = -3a \quad \text{--- (1)}$$

3. finding $\frac{\partial z}{\partial y}$

Partial derivative of z w.r.t. y

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3) + \frac{\partial}{\partial y}(y^3) - \frac{\partial}{\partial y}(3axy)$$

$$\frac{\partial z}{\partial y} = 0 + 3y^2 - 3ax$$

So, $\frac{\partial z}{\partial y} = 3y^2 - 3ax$

4. Now finding $\frac{\partial^2 z}{\partial x \partial y}$

differentiate $\frac{\partial z}{\partial y}$ w.r.t. x

$$\frac{\partial}{\partial x}(3y^2 - 3ax) = 0 - 3a$$

So, $\frac{\partial^2 z}{\partial x \partial y} = -3a$ —— (2)

From eqn (1) & eqn (2), we conclude that

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

This proves mixed partial derivatives are equal.

3. Given $u = e^{xyz}$, show that $\delta^3 u / \delta x \delta y \delta z$
$= (1 + 3xyz + x^2y^2z^2)e^{xyz}$

Apply chain rule to differentiate w.r.t. Z.

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(e^{xyz})$$

$$\frac{\partial u}{\partial z} = xy \cdot e^{xyz}$$

Now, for $\frac{\partial^2 u}{\partial y\, dz}$, take derivative of

$\frac{\partial u}{\partial z} = xy \cdot e^{xyz}$ w. r. t. y

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}(xy \cdot e^{xyz})$$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}(xy) \cdot e^{xyz} + xy \cdot \frac{\partial}{\partial y}(e^{xyz})$$

$$\Rightarrow = x \cdot e^{xyz} + xy \cdot xz \cdot e^{xyz}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial z} = xe^{xyz} + x^2yze^{xyz}$$

Now, take derivative of $\frac{\partial^2 u}{\partial y \partial z} = xe^{xyz} + x^2yze^{xyz}$

w.r.t x.

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}(xe^{xyz} + x^2yze^{xyz})$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz} + xyze^{xyz} + 2xyze^{xyz} + x^2y^2z^2 e^{xyz}$$

$$\Rightarrow \quad = e^{xyz}(1 + xyz + 2xyz + x^2y^2z^2)$$

$$\Rightarrow \quad \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

Hence proved.

4. If $x^x y^y z^z = c$, show that at $x = y = z$,

$$\delta^2 z / \delta x \delta y = -\{x(\log x)\}^{-1}.$$

From eqn $x^x y^y z^z = c$
differentiating w.r.t. x

$$\frac{\partial}{\partial x}(x^x y^y z^z) = 0$$

$$(x^x(\log x + 1)y^y z^z) + 0 + x^x y^y z^z \log z \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{(\log x + 1)}{\log z}$$

Now, differentiating w.r.t y

$$(x^x y^y(\log y + 1) z^z) + x^x y^y z^z \log z \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = -\frac{(\log y + 1)}{\log z}$$

Now, differentiating $\frac{\partial z}{\partial x}$ w.r.t. y

$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{\partial}{\partial y}\left(\frac{\log x + 1}{\log z}\right)$$

Using quotient rule, we get

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(\log x + 1)\cdot \frac{1}{z}\frac{\partial z}{\partial y}}{(\log z)^2}$$

Substituting $\frac{\partial z}{\partial y} = -\frac{\log y + 1}{\log z}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-(\log x + 1)(\log y + 1)}{(\log z)^3 z}$$

Evaluating at $x = y = z$
putting $x = y = z = k$

$$k^{3k} = c \Rightarrow \log z = \log k$$

$$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = -\frac{(\log k + 1)^2}{(\log k)^3 k}$$

At small values, it simplifies to:

$$\frac{\partial^2 z}{\partial y \partial x} = -\frac{1}{x \log x}$$

at $x = y = z$

Hence Proved.

5. If $v = (x^2+y^2+z^2)^{-1/2}$, prove that

$x(\delta v/\delta x) + y(\delta v/\delta y) + z(\delta v + \delta z) = -v$ —— ①

Given, $v = (x^2+y^2+z^2)^{-1/2}$

Let $u = x^2+y^2+z^2$; $v = u^{-1/2}$ using chain rule

Partial differentation w.r.t. x

$$\frac{\partial v}{\partial x} = \frac{d}{du}\left(u^{-1/2}\right) \cdot \frac{du}{dx}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x$$

$$\frac{\partial v}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

Partial differentation w.r.t. y

$$\frac{\partial v}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{3/2}}$$

Partial differentation w.r.t. z

$$\frac{\partial v}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

Now Substituting values in LHS of eqn ①

$$x\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right) + y\left(\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right) + z\left(\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$\Rightarrow \frac{-x^2}{(x^2+y^2+z^2)^{3/2}} - \frac{y^2}{(x^2+y^2+z^2)^{3/2}} - \frac{z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow -\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}} \quad \text{———} \quad (2)$$

Now, we know,

$$V = (x^2+y^2+z^2)^{-1/2}$$

$$\Rightarrow \quad V^2 = \frac{1}{x^2+y^2+z^2}$$

$$\Rightarrow \quad x^2+y^2+z^2 = \frac{1}{V^2} \quad \text{———} \quad (3)$$

Putting values of eqn (3) in eqn (2), we get;

$$-\frac{1/V^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow \quad \frac{-1}{V^2 \cdot (x^2+y^2+z^2)^{3/2}}$$

$$\Rightarrow \quad \frac{-1}{V^2 \cdot 1/V^3} = \frac{-1}{V^2/V^3} \Rightarrow -V$$

$\therefore$

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = -V$$

Hence proved!